Left modular and extremal lattices through brick-directed algebras

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FPSAC'26: U. Washington, Seattle
The 38th international conference (July 13-17, 2026) on Formal Power Series and Algebraic Combinatorics

Abstract:

We study some properties of semidistributive lattices that arise as the lattice of torsion classes of finite dimensional algebras. This includes some interesting finite and infintie lattices. More specifically,

1) First, we give a **realization of the left modular elements in terms of brick-splitting torsion classes.** Restricted to finite lattices, this immediately yields a full characterization of those lattices of torsion classes that are left modular (equivalently, extremal), thus trim.

2) Beyond the finite lattices, we further treat completely semidistributive weakly atomic lattices. In this setting we ultimately **generalize the notions of left modularity and extremality**.

Finally, through representation theory, we give an explicit construction to create **arbitrarily large left modular and extremal semidistributive lattices**. Moreover, we have developed the FDB Applet, as a useful computational tool to verify these properties for the lattice of torsion classes of gentle algebras and string algebras.

Representation Theory: Setting, Notations & *Definitions

k: a field **A**: a finite dimensional associative k-algebra **mod-A**: catergory of finite dim. left A-modules

ind(A): set of all isoclasses of indecomposables in mod-A. *A is **representation-finite** if ind(A) is a finite set.

*M in mod-A is a **brick** if $\mathrm{Hom}_A(M,M)$ is a division algebra (*i.e, every nonzero homomorphism f: M ---> M is invertible*).

brick(A): set of isoclasses of bricks in mod-A. *A is called **brick-finite** if brick(A) is a finite set.

*A full subcategory $\mathcal{T}$ is a **torsion class** if it is closed under quotients and extensions. More explicitly:

- If if $X \in \mathcal{T}$, any quotient module of X is also in $\mathcal{T}$.
- For each short exact sequence $0 \to L \to M \to N \to 0$ in mod A, if L and N are in $\mathcal{T}$, then so is M.

tors(A): set of all torsion classes in mod-A.

Remark: Torsion free classes are defined dually.
In particular, for $\mathcal{T}$ in tors(A), the corresponding torsion free class is $\mathcal{T}^{\perp} := \{X \in \text{mod } A \mid \mathrm{Hom}(T, X) = 0, \text{ for each } T \in \mathcal{T}\}$.

Lattice Theory: Setting, Notations & *Definitions

(L, ≤): a (possibly infinite) complete lattice, with **0** as minimum and **1** as maximum.

* L is **distributive** if, for every x, y, and z in L, we have
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

* L is **semidistributive** if, for every x, y, and z in L, we simultaneously have
- *Join semi-distributivity*: if $x \vee y = x \vee z$, then $x \vee y = x \vee (y \wedge z)$
- *Meet semi-distributivity*: if $x \wedge y = x \wedge z$, then $x \wedge y = x \wedge (y \vee z)$.

* L is **Completely semidistributive** if, for each x in L, and subset $S \subseteq L$, we have,
- If $x \wedge y = x \wedge z$ for all $y, z \in S$, then $x \wedge (\bigvee S) = x \wedge y$, for every $y \in S$
- If $x \vee y = x \vee z$ for all $y, z \in S$, then $x \vee (\bigwedge S) = x \vee y$, for every $y \in S$

* L is **weakly atomic** if, for all $x < y$, there exist u and v with $x \leq u \lessdot v \leq y$, where $u \lessdot v$ is a *cover relation (i.e, there is no element* z *with* $u < z < v$).

* $x \in L$ is **join-irreducible** if, for each y, z in L with $y, z < x$, then x is not $y \vee z$

* $x \in L$ is **completely join-irreducible** if x is not the join of $\{y \in L \mid y < x\}$.

$\mathrm{JI}^c(L)$: set of completely join-irreducible elements in L.

Remark: **(1)** The notion of (completely) meet-irreducible and $\mathrm{MI}^c(L)$ is defined dually.
(2) By convetion, 0 is not join-irreducible, and 1 is not meet-irreducible.

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Left modular Lattices & Extremal Lattices & Trim Lattices

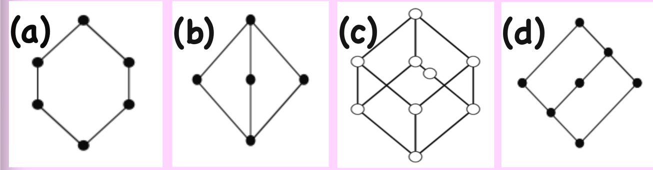
* $x \in L$ is **left modular** if, for each $y \leq z$ in L, we have $(y \vee x) \wedge z = y \vee (x \wedge z)$.

* A *finite* lattice is **left modular** if L has a maximal chain of left modular elements.

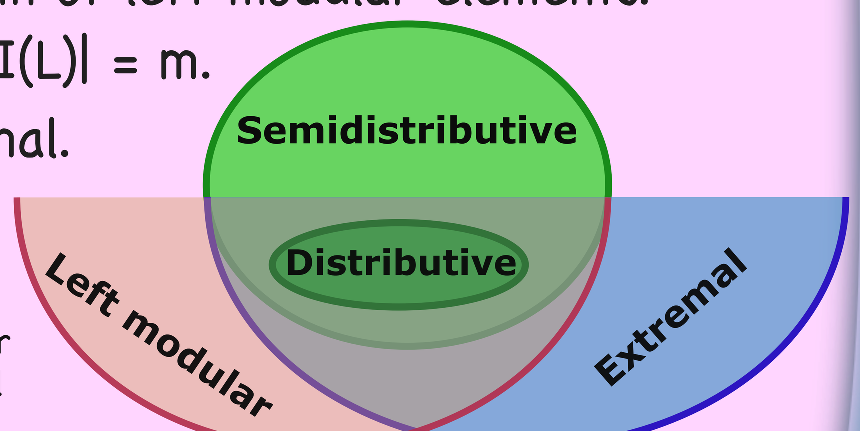
* A *finite* lattice of length m is **extremal** if $|\mathrm{JI}(L)| = |\mathrm{MI}(L)| = m$.

* A *finite* lattice is **trim** if it is left modular and extremal.

Remark: Finite distributive lattices are left modular and extremal, thus they are trim. The converse is false (see examples below).



(a) Semidistributive, not extremal/left modular
(b) Left modular, not semidistributive/extremal
(c) Extremal, not semidistributive/left modular
(d) Trim, not semidistributive



Theorem [Mu, TW]: For a finite semidistributive lattice L,
TFAE: **(1)** L is left modular **(2)** L is extremal **(3)** L is trim.

Remark: Semidistributivity is essential in the above theorem. See the above examples **(a)**, **(b)**, **(c)** and **(d)**.

Theorem [BCZ, DIJ, DIRRT, GM]: Let A be a finite dimensional algebra. Considered with respect to the inclusion-order, we have **tors(A) forms a** (not necessarily finite) **complete lattice which is completely semidistributive.**

Furthermore, **(1)** There exists a bijection between brick(A) and $\mathrm{JI}^c(L)$, and similarly between brick(A) and $\mathrm{MI}^c(L)$.
(2) Each cover relation $\mathcal{T} \lessdot \mathcal{T}'$ in tors(A) is determined by a unique brick.
(3) tors(A) is a finite lattice if and only if A is brick-finite.

Def[AIMP 1] A torsion class $\mathcal{T}$ is **brick-splitting** if every element of brick(A) belongs to $\mathcal{T}$ or to $\mathcal{T}^{\perp}$.

Theorem [AIMP 1]: **A torsion class $\mathcal{T}$ in mod-A is a left modular element of the lattice tors(A) $\Longleftrightarrow$ $\mathcal{T}$ is brick-splitting.**

Def[AIMP 1] A **brick path** in mod-A is a sequence of nonzero and non-isomorphism maps between elements in brick(A).
A **brick cycle** in mod-A is a brick path in which the first and last bricks are isomorphic.
A is called a **brick-directed algebra** if there is no brick cycle in mod-A.

Remark: (1) These notions are, respectively, the *brick-analogues* of the notions of *path* in mod-A, *cycle* in mod-A, and *representation-directed algebra*.
(2) Brick-directed algebras generalize rep-directed algebras. In fact, ***a brick-directed algebra can even be brick-infinite*** (see **Prop-Example** below).

Theorem [AIMP 1]: **Let A be a brick-finite algebra. Then, tors(A) is a left modular (equiv. extremal) lattice $\Longleftrightarrow$ A is brick-directed.**

Corollary: For a brick-finite algebra A, the lattice tors(A) is trim $\Longleftrightarrow$ there is a maximal chain of length |brick(A)| of brick-splitting torsion classes.

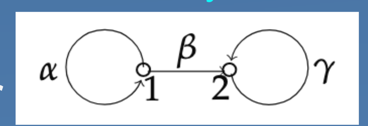
Def[AIMP 2]: Let L be a (possibly infinite) complete lattice. Then, we say
(1) L is **left modular** if it has an inclusion-maximal chain of left modular elements.
(2) L is **extremal** if there exists a bijection $\lambda: \mathrm{JI}^c(L) \to \mathrm{MI}^c(L)$ and a maximal chain X in L with the following propeties:
- For each $x \in X$, if $j \in \mathrm{JI}^c(L)$ satisfies $j \not\leq x$, then $x \leq \lambda(j)$.
- If $x \lessdot y$ is a cover relation for x, y in the chain X, then there is a unique completely join-irreducible j such that $j \leq y$ but $j \not\leq x$, and a unique completely meet-irreducible m such that $x \leq m$ but $y \not\leq m$, and $\lambda(j) = m$.

Remark: **(1)** Although the definition of extremality is rather technical, it is significantly more tractable in the setting of our next theorem.
(2) Restricted to finite lattices, our notions of left modularity and extremality conicide with the classical notions recalled earlier.

Theorem [AIMP 2]: Let L be a completely semidistributive weakly atomic lattice. Then,
(1) L is left modular if and only if L is extremal
(2) The set of all left modular elements in L form a completely distributive sublattice of L.

Cor: For each infinite cardinality $\aleph$, there is a left modular and extremal lattice with $2^{\aleph}$ many maximal chain of size $\aleph$ consisting of left modular elements.

Prop-Example:
Let **Q** be the quiver



(1) Let A=kQ/I, with I generated by αα=0=γγ. Then, tors(A) is a infinite, left modular & extremal lattice.
(2) Let A=kQ/J, with J generated by αα=βα=γγ=γβ=0. Then, tors(A) is a finite trim lattice.

Non-Example: Let A=kQ/I be the algebra given by $\bullet \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} \bullet$ with $\alpha_2\alpha_1 = 0$ Then, tors(A) is finite but not a trim lattice.

Remark/Applet: For rep-finite string algebra A, our **FDB Applet** is an efficient tool for computation of tors(A) and its lattice-theoretical properties.

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[AIMP 1 & AIMP 2] S. Asai, O. Iyama, K. Mousavand, C. Paquette (*arXiv:2506.13602* & *arXiv:2604.20947*). **[FDB Applet]** K. Mousavand. *Aailable on https://kavehmousavand.wixsite.com/math* (June 1, 2026).