On E-Symmetric Algebras and brick-Brauer–Thrall Conjectures

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A supervised AI-investigation on E-symmetric algebras, stable families, and brick-Brauer–Thrall Conjectures

Supervision, attribution, and declaration. All investigations summarized in this report were carried out under the supervision and direction of Kaveh Mousavand. Although KM has reviewed the content of this manuscript, edited, polished and improved different arguments for better readability and accuracy, he makes no claim of novelty, priority, or ownership over any new statement recorded here. As the section names indicate, the following results/observations were obtained at three different occasions, October 14th, 2026 (with a single prompt with GPT-6 Sol), November 11th, 2026 (with a single prompt with GPT-6 Sol), and November 12th, 2026 (with 3 consecutive prompts with GPT-6 Sol).

At the time of releasing this manuscript (November 16th, 2026), the sole purpose of this note is to document some observations and deductions that can provide new impetus to the bigger project and several problems listed in Brick Lodge repository, with attribution to the publicly available literature. Results quoted from the literature are distinguished from arguments supplied in this note. KM is primarily interested in the results on the brick-Brauer–Thrall (bBT) Conjectures, which are the outcome of the supervised research on November 12th and shortlisted below. In putting together this draft, KM has made every effort to provide correct and accurate attributions, and he welcomes feedback and suggestions to fix inaccuracies or errors, and to improve the results, exposition, and attributions.

Notation and Conventions. Throughout, k is an algebraically closed field and $A = kQ/I$ is a basic, connected, finite-dimensional unital associative k-algebra. Modules are finite-dimensional left modules, up to isomorphism, except when generic modules are explicitly considered. The products of paths are composed from right to left. Write S_i, $P_i = Ae_i$, and $I_i = D(e_iA)$ for the simple, indecomposable projective, and indecomposable injective modules at i, and $D = \mathrm{Hom}_k(-,k)$. A module B is a *brick* if $\mathrm{End}_A(B) \cong k$. A nonzero module B is θ-*stable* if $\theta(\underline{\dim}B) = 0$ and $\theta(\underline{\dim}L) < 0$ for every nonzero proper submodule $L \subset B$; see [Ki]. For all the notions that are not expelicitly define here, see [MP2, MP4, MP6] and the references therein.

Following [BS1], an algebra A is called E-*symmetric* if, for every $M \in \mathrm{mod}\, A$, we have

$$\dim_k \mathrm{Hom}_A(M, \tau M) = \dim_k \mathrm{Hom}_A(\tau^{-1}M, M).$$

For the motivations, background, and important properties of E-symmetric algebras, see [BS1, BHS2], and the references therein.

October 14th, 2026 (On characterizations and properties of E-symmetric algebras)

The earlier investigation studied the notion of an E-symmetric algebra introduced in [BS1], especially its interaction with brick infinitude, stable families, generic bricks, and Hom-orthogonality.

1.1 Characterization and Cartan identities

Using minimal projective and injective presentations, put

$$g_M = [P_0] - [P_1], \qquad h_M = [I^0] - [I^1].$$

Here g_M and h_M are regarded as column vectors in $\mathbb{Z}^{Q_0}$, using the projective and injective bases, respectively. Let q_{ij} be the number of arrows $i \to j$, and set

$$B_Q = (q_{ij} - q_{ji})_{i,j}, \qquad C_A = (\dim_k e_iAe_j)_{i,j}.$$

Theorem 1.1 (Alternative characterization)**.** *With the same notation as above, the following are equivalent:*

(i) *A is E-symmetric;*

(ii) *$g_M - h_M = B_Q\, \underline{\dim}M$, for every M.*

Remark (Added on November 11th). The above equivalence is the same as [BHS2, Proposition 3.13], after reversing the signs of both presentation vectors used there.

Corollary 1.2. *For an E-symmetric algebra A, one also obtains*

$$C_A^{\mathsf{T}} B_Q C_A = C_A^{\mathsf{T}} - C_A.$$

Indeed, put $H = (h_{P_1}, \ldots, h_{P_n})$. The first identity gives $I_n - H = B_Q C_A$. By [BS1, Section 1.4], A is known to be 1-Iwanaga–Gorenstein, hence the injective presentations of its projectives are resolutions, so $C_A^{\mathsf{T}} H = C_A$. If C_A is invertible over $\mathbb{Q}$, this yields

$$B_Q = C_A^{-1} - C_A^{-\mathsf{T}}.$$

1.2 Infinite semibricks of E-symmetric algebras

One has the following bound on the size of rigid bricks appearing in the semibricks of E-infinite algebras.

Theorem 1.3 (A quantitative rigid-semibrick bound)**.** *Let A be E-symmetric of rank n, and suppose that A is not strictly wild. Then, every semibrick contains at most $9n$ rigid bricks of finite projective dimension.*

Proof sketch: The finite-star argument in [MP5, proof of Proposition 3.2] bounds the number of distinct incoming and outgoing Ext^1-neighbours of any semibrick member by four each. The undirected Ext graph therefore has degree at most eight. Nine-colour any finite subset of the rigid members with finite projective dimension. In each colour class, the direct sum is rigid and has projective dimension at most one, hence is τ-rigid and has at most n summands. Finally, for a 1-Iwanaga–Gorenstein algebra, finite projective and injective dimension are equivalent.

Remark (Added on November 12th). Finite projective and finite injective dimension coincide for E-symmetric algebras [BHS2, §2.1]. Consequently, every infinite rigid semibrick has all but at most $9n$ members of both infinite projective and infinite injective dimension.

Note (November 13th): The single prompt on October 14th produced a 2-page document, part of which aimed at computational results on E-symmetric algebras of rank 2. That version is available upon request.

November 11th, 2026 (On a new realization of E-symmetric algebras)

The paper *E-symmetric algebras II* [BHS2] strengthens the structural picture through generalized Jacobian algebras and homological restrictions on minimal relations.

2.1 Generalized Jacobian characterization and relation bounds

The equivalences in [BHS2, Theorems 1.9, 4.1 and 5.5] characterize E-symmetric algebras by a simple-module condition, by the 1-Iwanaga–Gorenstein property together with the stated functorial duality, and by the generalized Jacobian property.

To avoid confusing relation multiplicities with Cartan entries, write

$$r_{ij} = \dim_k \mathrm{Ext}^2_A(S_i, S_j), \qquad I_i/S_i = X_i \oplus \bigoplus_j I_j^{\eta_{ij}},$$

where X_i has no injective direct summand. Then [BHS2, Theorem 4.1] gives

$$r_{ij} + \eta_{ij} = q_{ji}, \qquad r_{ij} \le q_{ji}.$$

The standard identification, recalled in [BHS2, Lemma 5.1], is

$$r_{ij} = \dim_k e_j \frac{I}{\mathfrak{m}I + I\mathfrak{m}} e_i,$$

where $\mathfrak{m}$ is the arrow ideal. Thus each independent minimal relation from i to j must be matched by an arrow from j to i.

2.2 Directed free gluing

Let Q° be obtained by deleting all loops from Q. If Q° is acyclic, then Q is called *weakly triangular* and have been treated in some other contexts; for example, see [AIMP, BZ], and the references therein. If Q is weakly triangular, the relation bound forces every minimal relation to be supported at a single vertex. Consequently, writing $R_i = e_i A e_i$ and $R = \prod_i R_i$, one obtains

$$A \cong T_R\left(\bigoplus_{\substack{a: i \to j \\ i \neq j}} R_j \otimes_k R_i\right).$$

There are no mixed defining relations. The local algebras R_i are E-symmetric by [BS1, Corollary 2.4], since here they are idempotent quotients. Conversely, generalized hyperpotentials for local algebras extend by zero on nonloop arrows; their compatibility maps extend blockwise.

Remark (Added on November 13th). We note that $R_i = e_i A e_i$ is a corner algebra, not generally an idempotent quotient. Under weak triangularity, however,$e_i A e_i \cong A/A(1-e_i)A$, so [BS1, Cor. 2.4] applies.

From the above argument, the generalized Jacobian characterization [BHS2, Theorem 5.5] therefore implies, the weakly triangular E-symmetric algebras are precisely the directed free gluings of local E-symmetric algebras. More precisely, for each vertex i, set $R_i = e_iAe_i$ and $R = \prod_{i\in Q_0} R_i$. And, for every nonloop arrow $a : i \to j$, let $M_a := R_j \otimes_k R_i$, with its natural R_j-R_i-bimodule structure. Finally, put

$$M_Q = \bigoplus_{\substack{a:i\to j \\ i\neq j}} M_a.$$

Theorem 2.1. *With the same notation as above, if Q is weakly triangular, the following are equivalent:*

(i) *A is E-symmetric*;
(ii) *all defining relations of A are local at vertices, and each R_i is E-symmetric*;
(iii) *$A \cong T_R(M_Q)$ and every R_i is E-symmetric.*

Consequently, if Q is acyclic, then A is E-symmetric if and only if $A = kQ$.

Proof sketch: The essential observation is the short connecting argument:

$$r_{ij} \le q_{ji} \Rightarrow r_{ij} = 0 \ (i \neq j) \Rightarrow \text{all minimal relations are local},$$

where $r_{ij} := \dim_k \operatorname{Ext}^2_A(S(i), S(j))$, and q_{ji} equal to the number of arrows $j \to i$. Then, for an E-symmetric algebra, [BHS2, Theorem 4.1] gives $r_{ij} + c_{ij} = q_{ji}$. Then, [BHS2, Theorem 4.1] supplies the relation bound and [BHS2, Theorem 5.5] the generalized-Jacobian characterization; [BS1, Corollary 2.4] supplies preservation under idempotent quotients. Note that, by [BHS2, Corollary 6.3], a local algebra is E-symmetric exactly when it is selfinjective and its unique nonprojective simple module, if there is one, satisfies $\Omega^4(S) \cong S$.

2.3 E-symmetric algebras as generalization of Jacobian algebras

Inspired by a recent counterexample to the Periodicity Conjecture by Enomoto [En], in Section 7 of [BHS2] the authors use the example to show symmetric algebras give a proper generalization fo Jacobian algebra. In particular, they consider the weakly symmetric algebras A_q on the Markov quiver. They are E-symmetric for $q \neq 0$, and are not classical Jacobian algebras when $q \neq 1$. The two-dimensional modules U_λ, with $\lambda \neq 0$, satisfy

$$\tau U_\lambda \cong U_\lambda \qquad \text{[BHS2, Corollary 7.7]}.$$

Their two nonzero parallel-arrow maps have the Kronecker form $1, \lambda$. Hence they are pairwise Hom-orthogonal bricks, stable for the weight which is 1 at the source, -1 at the target, and zero elsewhere. Thus A_q is brick-infinite and E-infinite; the latter also follows from [MP6, Theorem B]. Replacing λ by t gives a generic brick with endomorphism field $k(t)$ and endolength 2. Moreover,

$$\dim_k \operatorname{Ext}^1_{A_q}(U_\lambda, U_\lambda) = 1.$$

Indeed, Auslander–Reiten duality [ARS, Chapter IV, §4] identifies this space with the dual of the corresponding injectively stable Hom-space. Since U_λ is a noninjective brick and $\tau U_\lambda \cong U_\lambda$, no nonzero endomorphism can factor through an injective. Thus the stable Hom-space has dimension 1. This yields an infinite common-weight stable semibrick of nonrigid bricks.

Note (November 13th): The single prompt on November 11th produced a 3-page document, part of which aimed at computational results on E-symmetric algebras. That version is also available upon request.

November 12th, 2026 (On E-symmetric minimal brick-infinite algebras)

Let $\mathcal{F}_A$ denote the *support* of the τ-tilting fan. Thus A is g-tame if $\overline{\mathcal{F}_A} = \mathbb{R}^n$; it is E-finite if $\mathbb{Q}^n \subseteq \mathcal{F}_A$. Set

$$C = C_A, \qquad q_A(x) = x^{\mathsf{T}} C x, \qquad \widehat{x} = \frac{x}{\|x\|_1} \quad (x \neq 0).$$

The coordinates of g_M are always taken in the projective basis.

3.1 On stable second brick-Brauer–Thrall conjecture

Recall that A is *minimal brick-infinite* if A is brick infintie and every proper quotient of A is brick-finite. As discussed in details in [MP2], to verify several open conjecture on the study of brick, one can reduce them to minimal brick-infinite algebras; for details, see [MP2, B-L] and references therein. The following theorem verifies

the Stable Second brick-Brauer–Thrall Conjecture (and consequently several other open conjectures) for those minimal brick-infinite algebras which are E-symmetric. As remarked below, this does not settle the aforementioned conjectures for arbitrary E-symmetric algebras, yet it is an important observation and evidence towards these challenging problems.

Theorem 3.1 (Stable Second brick-Brauer–Thrall)**.** *Let A be E-symmetric and minimal brick-infinite. Then A is E-infinite. There exist a nonzero dimension vector $\mathbf{b} \in \mathbb{Z}^n_{\geq 0}$ and an integral weight $\theta \in K_0(\operatorname{proj} A)$ for which A has infinitely many pairwise nonisomorphic θ-stable bricks of dimension vector $\mathbf{b}$.*

One of the new tools in the systematic treatment of minimal brick-infinite algebras is the τ-convergence property, introduced in [MP4]. Briefly, this approach aims to construct a g-vector that admits an infinite family of indecomposable τ-rigid modules $\{X_i\}_{i\in\mathbb{N}}$ for which $\dim_k \operatorname{End}_A(X_i)$ grows subquadratically. In particular, to show that a minimal brick-infinite algebra A satisfies the 2nd bBT conjecture, it is sufficient (but not necessary) to show that A has the τ-convergence property; for details and examples, see [MP2, MP6] and references therein.

Theorem 3.2 (τ-convergence property)**.** *Let A be g-tame, E-symmetric, and minimal brick-infinite. There exist pairwise nonisomorphic indecomposable faithful τ-rigid modules T_j, with $\operatorname{pd}_A T_j = 1$, such that*

$$q_A(\widehat{g}_{T_j}) = \frac{\dim_k \operatorname{End}_A(T_j)}{\|g_{T_j}\|_1^2} \longrightarrow 0.$$

In particular, A has the τ-convergence property of [MP4, Definition 6.3], with an actual convergent sequence of indecomposable modules.

Remark. To compare the above results with the recent reductive result in [MP6, Corollary F] and [MP1, Theorem 1.4], note that neither theorem assumes that almost all bricks are faithful. In fact, the faithfulness input concerns only τ-rigid modules and is a consequence of minimal brick-infinitude.

3.2 Proofs of Theorem 3.1 and Theorem 3.2

Below, we present a proof, and provide detailed references to the results in the literature that are publicly available. In particular, the following play an important role and are used without reproving them:

Known inputs and crucial facts.

(i) By [DIJ, Theorem 1.4], brick-infinitude is equivalent to τ-tilting infinitude. Together with [AIR, Theorem 2.10], it implies that there are infinitely many indecomposable τ-rigid modules.

(ii) If A is E-symmetric, then $\dim_k \operatorname{Hom}_A(X, \tau X)$ is constant on each nonzero τ-orbit of indecomposables [BS1, §1.4, Remark (i)]. Thus every nonzero translate of an indecomposable τ-rigid module is again τ-rigid.

(iii) For a minimal brick-infinite algebra, only finitely many indecomposable τ-rigid modules are unfaithful, and faithful τ-rigid modules have projective dimension at most one [MP1, Theorem 4.2]. Only finitely many basic support τ-tilting modules fail to be classical tilting modules [MP1, Corollary 4.4].

(iv) The g-vector map is injective on isomorphism classes of τ-rigid pairs [AIR, Theorem 5.5]. For a τ-rigid module, the two projectives in its minimal presentation have no common nonzero summand [AIR, Proposition 2.5].

(v) The Cartan–Euler computation in [MP4, §6.1] gives

$$q_A(g_T) = \dim_k \operatorname{End}_A(T) > 0$$

for every nonzero τ-rigid module T of projective dimension at most one. The mixed-term calculation on a classical tilting cone is [MP4, Lemma 6.2, proof].

(vi) A minimal brick-infinite algebra with τ-convergence is E-infinite [MP4, Theorem 6.8]. An E-infinite algebra admits the common-weight stable family in Theorem 3.1 [MP6, Theorem A].

Let A be an E-symmetric minimal brick-infinite algebra. If A is not g-tame, the existence of the stable-family assertion follows immediately from [MP6, Corollary D]. In fact, the underlying implication to E-infiniteness is [MP4, Remark 6.5, first paragraph of its proof]. Therefore, to complete the proof of Theorem 3.1, it remains to prove Theorem 3.2, which treats g-tame E-symmetric minimal brick-infinite algebras.

Proof of Theorem 3.2: First, we develop some new tools and make some handy observations below.

Lemma 1 (Faithful adjacent translates)**.** *Let X be an indecomposable τ-rigid module such that X and τX are faithful. Then $\operatorname{Hom}_A(X, A) = 0 = \operatorname{Hom}_A(DA, \tau X)$, and there is an isomorphism of k-algebras*

$$\operatorname{End}_A(X) \cong \operatorname{End}_A(\tau X).$$

Proof. For every faithful finite-dimensional module M, evaluation on a k-basis of M gives an embedding $A \hookrightarrow M^a$. Applying the same observation to the faithful right module DM and dualizing gives an epimorphism $M^b \twoheadrightarrow DA$. Using $M = \tau X$ for the embedding and $M = X$ for the epimorphism, the equality $\mathrm{Hom}_A(X, \tau X) = 0$ gives the two asserted vanishings.

Every map from X through a projective is therefore zero, and every map into τX through an injective is zero. The Auslander–Reiten equivalence between the projectively and injectively stable module categories [ARS, Chapter IV, §4] consequently identifies the *ordinary* endomorphism algebras:

$$\mathrm{End}_A(X) \cong \underline{\mathrm{End}}_A(X) \cong \overline{\mathrm{End}}_A(\tau X) \cong \mathrm{End}_A(\tau X).$$

□

Now, we consider two cases with respect to the τ-orbit of indecomposable τ-rigid modules, and accordingly treat the Theorem 3.2 under the assumption of g-tameness for the E-symmetric minimal brick-infinite algebra A.

Case 1: there exists an infinite τ-orbit. The translation τ is a partial bijection on indecomposable isomorphism classes, with inverse τ^{-1}. Thus an infinite orbit contains a one-sided sequence

$$X_j = \tau^{\varepsilon j} X_0, \qquad j \geq 0, \qquad \varepsilon \in \{1, -1\},$$

of pairwise nonisomorphic modules. A repetition would produce a finite cycle, and injectivity of the partial translation precludes an additional tail attached to that cycle.

If the orbit contains a τ-rigid module, all its members are τ-rigid by input (ii). The finite exceptional set in input (iii) meets the sequence in only finitely many positions. Delete the initial segment through the last such position. Delete a further finite initial segment to avoid all projective and injective terms. The resulting tail (T_j) consists of faithful indecomposable τ-rigid modules of projective dimension one.

Lemma 1 applies to every adjacent pair, in either direction along the tail. Hence $\dim_k \mathrm{End}_A(T_j) = e$, for a fixed positive integer e. Their integral g-vectors are pairwise distinct by [AIR, Theorem 5.5]. Every bounded subset of $\mathbb{R}^n$ contains only finitely many lattice points, and each can occur at most once. Therefore, $\|g_{T_j}\|_1 \longrightarrow \infty$. Input (v) from [MP4, §6.1] now yields

$$q_A(\widehat{g}_{T_j}) = \frac{e}{\|g_{T_j}\|_1^2} \longrightarrow 0.$$

Remark: We note that this part of the argument does not use g-tameness.

Case 2: all τ-orbits of indecomposable τ-rigid modules are finite. There are infinitely many such modules by input (i) from [DIJ, Theorem 1.4]. Remove every orbit meeting an unfaithful module, a projective, or an injective. Only finitely many finite orbits are removed. A remaining finite orbit has no endpoint, hence is periodic:

$$T, \tau T, \ldots, \tau^{r-1} T, \qquad \tau^r T \cong T.$$

Every term is faithful and has projective dimension one. Put

$$U = \bigoplus_{i=0}^{r-1} \tau^i T, \qquad u = g_U, \qquad \mathbf{d} = \underline{\dim} U.$$

Then $\tau U \cong U$, with $\mathrm{pd}_A U \leq 1$ and $\mathrm{Hom}_A(U, A) = 0$, by Lemma 1. However, we do *not* assert that U is τ-rigid.

The projective resolution of U gives $Cu = \mathbf{d}$. Using the Nakayama functor $\nu = D\,\mathrm{Hom}_A(-, A)$, the standard sequence associated with the minimal projective presentation is

$$0 \longrightarrow \tau U \longrightarrow \nu P_1 \longrightarrow \nu P_0 \longrightarrow D\,\mathrm{Hom}_A(U, A) \longrightarrow 0;$$

see [AIR, Proposition 2.4(a)]. Since $\underline{\dim}(\nu P_i)$ is obtained from the projective multiplicity vector using C^{T}, we obtain

$$Cu = \mathbf{d}, \qquad C^{\mathsf{T}} u = -\mathbf{d}.$$

In particular,

$$0 \neq u \in \ker(C + C^{\mathsf{T}}) \cap \mathbb{Z}^n, \qquad q_A(u) = 0. \tag{1}$$

Why is this particular null vector outside the τ-tilting fan? To answer this question, we note that the isotropy $q_A(u) = 0$, by itself, is insufficient. Hence, we also use an injective presentation in class u. Write

$$0 \longrightarrow P_1 \xrightarrow{f} P_0 \longrightarrow U \longrightarrow 0$$

for a minimal projective resolution and decompose

$$P_1 = P(u^-) \oplus R, \qquad P_0 = P(u^+) \oplus R,$$

where $u = u^+ - u^-$ is the coordinatewise positive-negative decomposition. Note that one cannot simply cancel R from the given minimal differential. Instead, work in the affine space $\mathrm{Hom}_A(P_1, P_0)$.

Injectivity is a nonempty open condition in the space $\mathrm{Hom}_A(P_1, P_0)$, because f is injective. And, invertibility of the R-to-R block is another nonempty open condition in $\mathrm{Hom}_A(P_1, P_0)$; hence their intersection is nonempty open subset of $\mathrm{Hom}_A(P_1, P_0)$. It follows that the reduced presentation space

$$H_u = \mathrm{Hom}_A(P(u^-), P(u^+))$$

has a dense open injective locus.

We now apply some geometric arguments used and developed in [Pl, MP4] at the following precise points:

(a) By [Pl], generic cokernels in H_u belong to the τ-regular component $\mathcal{Z}(u)$. Intersect the generic presentation locus with the injective locus. The resulting generic cokernel V satisfies

$$\mathrm{pd}_A V \leq 1, \qquad g_V = u, \qquad \underline{\dim} V = Cu = \mathbf{d} \neq 0.$$

The presentation is minimal: its two projectives have disjoint indecomposable supports, so the differential is radical; injectivity identifies the source with the kernel.

(b) Membership $u \in \mathcal{F}_A$ would give $c(\mathcal{Z}(u)) = 0$, by the fan interpretation in [MP4, §6.3]. The discussion preceding Theorem 5.1 there identifies such components with orbit closures of τ-rigid modules. Consequently the generic V above would be τ-rigid.

(c) Since $\mathrm{pd}_A V \leq 1$ and V is τ-rigid, the Cartan–Euler calculation [MP4, §6.1] would then give

$$0 = q_A(u) = \dim_k \mathrm{End}_A(V) > 0,$$

which is a contradiction.

Therefore

$$u \notin \mathcal{F}_A. \tag{2}$$

Remark: The above argument does not assume that the original module U is generic in $\mathcal{Z}(u)$. Instead, it construct the generic module V with the desired properties.

Recovering indecomposable convergence. Let $\mathscr{E}$ be the union of the closed maximal cones corresponding to basic support τ-tilting modules which are not classical tilting modules. By [MP1, Corollary 4.4], this is a finite union of closed cones. Since $\mathscr{E} \subseteq \mathcal{F}_A$, equation (2) implies $u \notin \mathscr{E}$.

A neighborhood of u therefore avoids $\mathscr{E}$. By g-tameness assumption of the algebra A, choose nonzero

$$w_j \in \mathcal{F}_A \setminus \mathscr{E}, \qquad w_j \longrightarrow u.$$

Each w_j lies in a classical tilting cone. Write its active generators as $g_{T_{j1}}, \ldots, g_{T_{js_j}}$, where $s_j \leq n$. If we apply [AIR, Proposition 2.5] to their direct sum, we get that these vectors are sign-coherent. Hence normalization gives

$$\widehat{w}_j = \sum_{\ell=1}^{s_j} a_{j\ell} \widehat{g}_{T_{j\ell}}, \qquad a_{j\ell} \geq 0, \qquad \sum_{\ell} a_{j\ell} = 1.$$

The mixed terms are nonnegative by the calculation in [MP4, Lemma 6.2, proof]:

$$g_{T_{j\ell}}^{\mathsf{T}} C g_{T_{jm}} = \dim_k \mathrm{Hom}_A(T_{j\ell}, T_{jm}) \geq 0.$$

Therefore

$$\begin{aligned} q_A(\widehat{w}_j) &\geq \sum_{\ell} a_{j\ell}^2 q_A(\widehat{g}_{T_{j\ell}}) \\ &\geq \frac{1}{n} \min_{\ell} q_A(\widehat{g}_{T_{j\ell}}). \end{aligned}$$

Since $q_A(\widehat{w}_j) \to q_A(\widehat{u}) = 0$, a minimizing summand X_j satisfies $q_A(\widehat{g}_{X_j}) \to 0$.

For every fixed nonzero indecomposable partial tilting module X, [MP4, §6.1] gives

$$q_A(\widehat{g}_X) = \frac{\dim_k \mathrm{End}_A(X)}{\|g_X\|_1^2} > 0.$$

Consequently, for every finite collection $\mathscr{S}$ of such modules,

$$\delta_{\mathscr{S}} := \min_{X \in \mathscr{S}} q_A(\widehat{g}_X) > 0.$$

Once $q_A(\widehat{g}_{X_j}) < \delta_{\mathscr{S}}$, the chosen summand is not in $\mathscr{S}$. The sequence therefore eventually avoids every finite collection, and a pairwise nonisomorphic subsequence can be selected. This uses positivity on each *fixed* ray, not a uniform positive lower bound on all partial tilting rays. Finally remove the finitely many unfaithful and projective terms, using [MP1, Theorem 4.2]. This proves Theorem 3.2.

Proof of Theorem 3.1. The non-g-tame case was handled above; see [MP6, Corollary D]. In the g-tame case, Theorem 3.2 and [MP4, Theorem 6.8] give E-infiniteness. The stable family follows from [MP6, Theorem A]. □

Remark. The above results on E-symmetric algebras, particularly Theorems 3.1 and 3.2, immediately imply that some other open brick-Brauer–Thrall conjectures hold for the algebras under consideration. For more details, see [MP2, MP6, B-L], and there references therein.

3.3 Examples & Non-examples

To put the scope of the above results on E-symmetric minimal brick-infinite algebras into a better perspective, we collect some explicit examples and supplementary observations which clarify the necessity and sufficiency of the assumptions in Theorems 3.1 and 3.2.

A uniform verification of E-symmetry. For the finite-dimensional algebras below whose defining relations are powers $\varepsilon^{m_\varepsilon}$ of individual loops, take the hyperpotential

$$\rho_\varepsilon = \varepsilon^{m_\varepsilon}, \qquad \rho_a = 0 \quad \text{for nonloop arrows.}$$

Its commutator sum is zero. Thus [BS1, Theorem 1.2] applies. This works in every characteristic; it does not require integration to a potential or division by $m_\varepsilon + 1$.

Example 3.3 (Affine hereditary algebras)**.** For every acyclic affine Dynkin quiver Q, namely, $\widetilde{A}_m, \widetilde{D}_m, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, the path algebra kQ is E-symmetric and minimal brick-infinite. The fact that these algebras are E-symmetric is included in [BS1, Proposition 1.4].

Example 3.4 (Double-loop generalized barbell algebras)**.** Take a path on vertices $v_0, \dots, v_m$, with $m \geq 1$, in any orientation. Add a loop α at v_0 and a loop β at v_m, and impose only the quadratic relations $\alpha^2 = \beta^2 = 0$. These generalized barbell algebras (which are gentle) are minimal brick-infinite; see [MP1, Remark 4.3], and the generalized-barbell construction in [MP3, §2.4]. Their E-symmetry follows from the hyperpotential above. We remark that, unlike the hereditary case in 3.3, the generalize barbell algebras are not necessarily minimal representation-infinite.

Example 3.5 (A square-zero loop between a source and a sink.)**.** Let

$$Q_1 = \{a : 1 \to 2,\ b : 2 \to 3,\ \varepsilon : 2 \to 2\}, \qquad A = kQ/(\varepsilon^2).$$

This is the algebra $C(1)$ in the classification [Bo, Theorem 1]. Its minimal representation-infinitude comes from that theorem, and its brick-infinitude follows from [Mo, Theorem 1.2(2)], hence it is minimal brick-infinite. It is E-symmetric by the same hyperpotential construction as above.

Remark (**g-tameness and minimality without E-symmetry**). To emphasize on each assumption in the proof of Theorem 3.2, note that g-tameness and minimality without E-symmetry do not imply the τ-convergence; also see [MP6, Corollary F and Remark after that]. More specifically, in this context, for the minimal brick-infinite algebra Ω_3 considered in [Wa2, Proposition 3.8], the g-tameness is established in [MP6, Example 4.1]. Its Cartan matrix and quadratic form are

$$C_{\Omega_3} = \begin{pmatrix} 1 & 0 \\ 3 & 3 \end{pmatrix}, \qquad q_{\Omega_3}(x,y) = x^2 + 3xy + 3y^2 = \left(x + \frac{3}{2}y\right)^2 + \frac{3}{4}y^2.$$

This form is positive definite. Its minimum on the compact ℓ^1-unit sphere is therefore positive, so Ω_3 has no τ-convergence. Meanwhile, we note that this algebra is not E-symmetric: its idempotent quotient at vertex 2 is the radical-square-zero local algebra with two loops, which is not selfinjective. Use [BS1, Corollary 2.4] and [BHS2, Corollary 6.3].

Remark (**E-symmetric algebras are not quotient closed**). As observed below, to settle the brick-Brauer–Thrall conjectures for all E-symmetric algebras, it is not sufficient to only treat those which are minimal brick-infinite. More specifically, there exist brick-infinite E-symmetric algebras with no E-symmetric minimal brick-infinite quotient. For example, take vertices s, t, v_1, v_2, v_3, arrows

$$a_i : s \to v_i, \qquad b_i : v_i \to t \quad (1 \leq i \leq 3), \qquad c : t \to s,$$

and put $p_i = b_i a_i$. Define

$$\Lambda = kQ \Big/ \big(p_1 + p_2 + p_3,\ a_i c,\ c b_i \mid 1 \le i \le 3\big).$$

This is the Jacobian algebra of

$$W = c(p_1 + p_2 + p_3).$$

All paths of length three vanish; there are five vertices, seven arrows, and two independent length-two classes. Hence, $\dim_k \Lambda = 14$. It is E-symmetric by [BS1, Theorem 1.2].

The quotient algebra

$$B = \Lambda/(c) = k\big(s \xrightarrow{a_i} v_i \xrightarrow{b_i} t\big)_{i=1}^{3} \Big/ (p_1 + p_2 + p_3)$$

is the domestic canonical algebra of weight type $(2, 2, 2)$. It is minimal representation-infinite by [Bo, Theorem 1], and brick-infinite by [Mo, Theorem 1.2(2)]. Thus B, and consequently Λ, is brick-infinite.

The algebra B is not E-symmetric: it has one minimal relation from s to t and no arrow from t to s, so

$$r_{st} = 1 > 0 = q_{ts},$$

contrary to [BHS2, Theorem 4.1 and Lemma 5.1].

Finally let $\Gamma = \Lambda/J$ be a minimal brick-infinite quotient. If the image of c is nonzero, then s, t survive and

$$P_t^{\Gamma} \cong I_s^{\Gamma}.$$

Both are the length-two representation on $t \xrightarrow{c} s$: the relations annihilate every other nontrivial path starting at t or ending at s. This would give a nonzero projective-injective module, contradicting [MP1, Proposition 3.1]. Thus $c \in J$ and Γ is a quotient of B. Minimal brick-infinitude of B forces $J = (c)$.

In summary, we conclude that Λ is E-symmetric, B is its unique minimal brick-infinite quotient, and B is not E-symmetric. Accordingly, Theorem 3.1 cannot be extended to every E-symmetric algebra merely by passing to a minimal brick-infinite quotient. Finally, note that the idempotent-quotient preservation theorem [BS1, Corollary 2.4] does not apply to this arbitrary quotient.

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Edition. This manuscript is the result of editions by Kaveh Mousavand applied to the AI-generated document. In particular, the content is adjusted, both in exposition and length, for better readability and citation accuracy. The original AI-generated document (generated on November 12th, 2026, by Chat GPT-6 Sol) is a few pages longer and contains some arguments that are not illuminating. Upon request, the original tex file is available.

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Remark. A pen and paper analysis by KM on November 13th, followed by a separate prompt to Chat GPT-6 Sol for verification of his main ideas, suggests that some of the results on E-symmetric minimal brick-infinite algebras can be further generalized. This particularly strengthens the applications to the brick-Brauer–Thrall Conjectures beyond the setting considered above. More specifically, one can consider the *weakly minimal brick-infinite algebras*, formulated by KM in his PhD dissertation, Notably, this includes the famous list of algebras independently obtained by Bongartz and Happel–Vossieck in the 1980's, as "minimal representation-infinite algebras", where that notion of minimality considered is different from the more common one in the recent literature and it is only with respect to the proper quotient by ideals generated by an idempotent element. Analogously, it is evident that every minimal brick-infinite algebra is weakly minimal brick-infinite, but not the converse. In contrast with the above Remark on "E-symmetric algebras are not quotient closed", from [BS1, BHS2] it is immediate that brick-infinite E-symmetric algebras admit E-symmetric quotients which are E-symmetric weakly minimal brick-infinite. Hence, such a reduction can provide more tools in the study of bBT Conjectures for all E-symmetric algebras. After further verifications and possible strengthening of the aforementioned new results on E-symmetric weakly minimal brick-infinite algebras by KM, they may be added to this document or posted in a separate manuscript.

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